

Sampling theorem:

Statement:

A continuous time signal can be completely represented in its samples and recovered back if the sampling frequency is twice of highest frequency content of the signal i.e.,

$$f_s \geq 2\omega$$

Proof: Sampling: The sampled signal is given as,

$$x_s(t) = \sum_{n=-\infty}^{\infty} x(t) \delta(t - nT_s)$$

Taking Fourier transform of this signal,

$$\begin{aligned} X_s(f) &= FT \left\{ \sum_{N=-\infty}^{\infty} x(t) \delta(t - nT_s) \right\} \\ &= FT \{ x(t) \} * FT \{ \delta(t - nT_s) \} \\ &= x(f) * f_s \sum_{n=-\infty}^{\infty} \delta(f - n f_s) \\ &= f_s \sum_{n=-\infty}^{\infty} x(f) * \delta(f - n f_s) \\ &= f_s \sum_{n=-\infty}^{\infty} x(f - n f_s) \longrightarrow \textcircled{1} \end{aligned}$$

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Above equation shows that $x(f)$ is placed at $\pm f_s, \pm 2f_s, \pm 3f_s$

FT of DT signal is given as,

$$x(f) = \sum_{n=-\infty}^{\infty} x(n) e^{-j2\pi f n}$$

Here $x(f)$ is continuous and f is continuous frequency. If we replace $x(f)$ by $X_S(f)$, then will be replaced by $\frac{f}{f_s}$ continuous frequency

$$\therefore X_S(f) = \sum_{n=-\infty}^{\infty} x(n) e^{-j2\pi \frac{f}{f_s} n}$$

Here note that ' f ' is continuous frequency and $\frac{f}{f_s}$ is discrete frequency. Also $x(n) = x(nT_s)$ & $\frac{1}{f_s} = T_s$

$$\therefore X_S(f) = \sum_{n=-\infty}^{\infty} x(nT_s) e^{-j2\pi f n T_s} \rightarrow (2)$$

$$\text{If } f_s = 2\omega$$

then from equation (1) we can write

$$X_S(f) = f_s X(f) \quad \text{for } -\omega \leq f \leq \omega$$

$$\omega \quad f_s = 2\omega$$

$$\text{or } X(f) = \frac{1}{f_s} X_S(f)$$

Putting for $X_S(f)$ from equation (2),

$$X(f) = 1/f_s \sum_{n=-\infty}^{\infty} x(nT_s) e^{-j2\pi f n T_s}$$

Taking inverse fourier transform of above equation,

$$x(t) = \int_{-\infty}^{\infty} \left[1/f_s \sum_{n=-\infty}^{\infty} x(nT_s) e^{-j2\pi f n T_s} \right] e^{j2\pi f t} \cdot df$$

Above equation (3) shows that $x(t)$ is completely represented in its samples $x(nT_s)$

Recovery :

Since,

$$X(f) = 1/f_s X_s(f) \text{ for } -\omega \leq f \leq \omega$$

equation (3) we can written as,

$$x(t) = \int_{-\omega}^{\omega} 1/f_s \sum_{n=-\infty}^{\infty} x(nT_s) e^{-j2\pi f n T_s} \cdot e^{j2\pi f t} \cdot df$$

$$= \sum_{n=-\infty}^{\infty} x(nT_s) \cdot 1/f_s \int_{-\omega}^{\omega} e^{j2\pi f (t - nT_s)} df$$

$$= \sum_{n=-\infty}^{\infty} x(nT_s) / f_s \left[\frac{e^{j2\pi f (t - nT_s)}}{j2\pi (t - nT_s)} \right]_{-\omega}^{\omega}$$

$$= \sum_{n=-\infty}^{\infty} x(nT_s) \cdot 1/f_s \left\{ \frac{e^{j2\pi \omega (t - nT_s)} - e^{-j2\pi \omega (t - nT_s)}}{j2\pi (t - nT_s)} \right\}$$

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$$= \sum_{n=-\infty}^{\infty} x(nT_s) \cdot \frac{1}{f_s} \cdot \frac{\sin 2\pi\omega(t-nT_s)}{\pi(t-nT_s)}$$

$$= \sum_{n=-\infty}^{\infty} x(nT_s) \cdot \frac{\sin \pi(2\omega t - 2\omega nT_s)}{\pi(2\omega t - 2\omega nT_s)} \quad \text{Here } f_s = 2\omega$$

Here,

$$T_s = \frac{1}{f_s} = \frac{1}{2\omega} \quad \text{Hence } 2\omega T_s = 1$$

$$\therefore x(t) = \sum_{n=-\infty}^{\infty} x(nT_s) \cdot \frac{\sin \pi(2\omega t - n)}{\pi(2\omega t - n)}$$

$$= \sum_{n=-\infty}^{\infty} x(nT_s) \cdot \text{sinc}(2\omega t - n)$$

$$\text{Since } \frac{\sin \pi\theta}{\pi\theta} = \text{sinc}\theta$$

Here the samples $x(nT_s)$ are weighted by sinc function. Sinc function is interpolation function. Thus the signal $x(t)$ is completely recovered from its samples.

⇒ Aliasing :

The spectrum of sampled signal is given

$$X_s(f) = f_s \sum_{n=-\infty}^{\infty} X(f - n f_s)$$

Here observe that spectrum located at $x(f)$, $x(f \pm f_s)$... overlap on each other. The frequency from $f_s - \omega$ to ω are overlapping this show by shaded area.

Defination aliasing:

When the high frequency interferes with low frequency interferes with low frequency and appears as low frequency, it is called aliasing.

Ways to avoid aliasing:

i) Sampling theorem rate higher than $f_s > 2\omega$:

Having sampling rate higher than 2ω , the overlap between the spectrum of $x(f)$, $x(f \pm f_s)$, $x(f \pm 2f_s)$... is absent. Hence there is aliasing.

ii) Bandlimiting the signal:

Minimum sampling rate for avoiding aliasing is 2ω . Hence signal must be lowpass filtered to avoid any frequency components higher than 2ω . This avoids frequency

Nyquist rate & Nyquist interval:

Nyquist rate:

When the sampling rate become exactly equal to ' 2ω ' samples/sec, for a given bandwidth ω

ω Hertz, then it is called Nyquist rate.

Nyquist interval: It is the time interval between any two adjacent samples when sampling rate is Nyquist rate

$$\text{Nyquist rate} = 2\omega \text{ Hz}$$

$$\text{Nyquist interval} = \frac{1}{2\omega} \text{ seconds.}$$

⇒ Impulse Sampling:

Width of pulse approaches zero—train of impulses of produced. This method is not used, since a pulse of zero width of can't be produced.

Natural Sampling:

The pulse has finite width. The top of pulse takes the shape of original signal. The spectrum naturally sampled signal is weighted by sinc function

Rectangular pulse (or) flat top Sampling:

The pulse has finite width and its top flat.

This method is also used in practical application.

Autocorrelation function for energy signals:-

Defination:

When we calculate correlation function of the signal with itself, then it is called autocorrelation.

Thus if $x_1(t) = x_2(t)$, then correlation becomes autocorrelation.

Autocorrelation function of energy signals: We can write autocorrelation function for energy signals as,

$$R(z) = \int_{-\infty}^{\infty} x(t) x^*(t-z) dt$$

In this equation, the complex valued signal is delayed. In other words, the real valued signal is advanced. Hence above equation can also be written as,

$$R(z) = \int_{-\infty}^{\infty} x(t+z) x^*(t) dt$$

Similarly, autocorrelation function of discrete time signals given as,

$$R(m) = \sum_{n=-\infty}^{\infty} x(n) x^*(n-m)$$

$$= \sum_{n=-\infty}^{\infty} x(n+m) x^*(n)$$

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Auto correlation for power signals:

The autocorrelation function for periodic signals can be written as,

Autocorrelation for periodic signals given by

$$R(z) = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t) x^*(t-z) \cdot dt$$

if z is in negative direction then,

$$R(z) = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t+z) x^*(t) \cdot dt$$

or for any period,

$$R(z) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t) x^*(t-z) \cdot dt$$

Here note that T_0 is period of signals. The autocorrelation function for discrete time power signals can be expressed.

$$R(m) = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=-N}^N x(n) x^*(n+m)$$

Relation between auto correlation of energy & power of Signals:

We defined auto correlation function energy signals and power signals. Now let us related auto correlation to energy & power of signal.

Relation of auto correlation with energy - Consider the auto correlation of energy signals given by

$$R(z) = \int_{-\infty}^{\infty} x(t) x^*(t-z) \cdot dt$$

With zero shift ($z=0$) above eq'n will be

$$R(0) = \int_{-\infty}^{\infty} x(t) x^*(t) \cdot dt = \int_{-\infty}^{\infty} |x(t)|^2 \cdot dt$$

$$= E \quad \therefore \text{by definition}$$

$$R(0) = E$$

Thus at ' $z=0$ ' auto correlation is equal to total energy of signal. Above relation also holds for discrete time signal.

Relation of auto correlation with power: Consider the auto correlation of power signal given by eq'n

$$R(z) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t) x^*(t-z) \cdot dt$$

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With zero shift ($z=0$) above equation becomes,

$$R(0) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t) x^*(t) \cdot dt$$

$$= \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 \cdot dt$$

= P by definition of power

$$R(0) = P$$

Thus at $z=0$, autocorrelation is equal to total power of these signal. Above relation is also applicable for discrete time signals.

Auto correlation properties:

Property 1: The autocorrelation function shows conjugate

Symmetry i.e.,

$$R(z) = R^*(-z)$$

Proof:

$$R(z) = \int_{-\infty}^{\infty} x(t) x^*(t-z) \cdot dt$$

$$= E \quad \therefore \text{by definition}$$

$$R(0) = E$$

(6)

Thus at $z=0$, auto correlation is equal to total energy of the signal. Above relation also holds for discrete time signals.

$$R^*(z) = \int_{-\infty}^{\infty} x^*(t) x(t-z) \cdot dt$$

by taking complex conjugate

$$\therefore R^*(-z) = \int_{-\infty}^{\infty} x^*(t) x(t+z) \cdot dt$$

$$= R(z)$$

$$\therefore R(z) = R^*(-z)$$

Property 2-8

The value of auto correlation of function of $z=0$ is equal to energy of the signal

$$R(0) = E = \int_{-\infty}^{\infty} |x(t)|^2 \cdot dt$$

Proof $R(z) = \int_{-\infty}^{\infty} x(t) x^*(t-z) \cdot dt$

put $z=0$

$$R(0) = \int_{-\infty}^{\infty} x(t) x^*(t) \cdot dt = \int_{-\infty}^{\infty} |x(t)|^2 \cdot dt$$

$$= E$$

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Property-3

If τ is increased in either direction, the auto correlation reduces, As τ reduces, auto correlation increases & maximum at $\tau=0$ i.e. at origin i.e. ...

$$|R(\tau)| \leq R(0) \text{ for all } \tau$$

$$\text{i.e. } x^2(t) + x^2(t+\tau) \pm 2x(t)x(t+\tau) \geq 0$$

$$x^2(t) + x^2(t+\tau) \geq \pm 2x(t)x(t+\tau)$$

Integrating both sides,

property: 4

The autocorrelation function and energy spectral density function of energy signal forms a fourier transform pair i.e.,

$$R(\tau) \longleftrightarrow Y(f) \rightarrow \text{④}$$

proof: The property will be proved in the next section. It requires knowledge of ESD.

property: 5

The autocorrelation function & power spectrum density form a fourier transform pair i.e.,

$$\boxed{R(\tau) \longleftrightarrow S(f)} \rightarrow \text{④}$$

proof: This property is proved in the next section. It requires the knowledge of PSD.

Cross correlation functions

- * correlation between two different signals is called cross correlation.
- * cross correlation function exhibits conjugate symmetry.
- * The correlation of two signals is obtained by convolution of one signal with time folded version of another signal.

$$R_{12}(\tau) = x_1(-\tau) * x_2(\tau) \text{ (or)}$$

$$R_{12}(\tau) = x_1(-\tau) * x_2(\tau) ,$$

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Define cross correlation of energy signals.

→ The two signals $x_1(t)$ & $x_2(t)$ is a pair of complex valued signals of finite energy. The cross correlation of these two signals is given as,

Cross correlation of energy signals:

$$R_{12}(\tau) = \int_{-\infty}^{\infty} x_1(t) x_2^*(t-\tau) dt$$

List the properties of cross correlation function.

Define cross correlation of periodic signals.

→ The cross correlation function $R_{12}(\tau)$ for two periodic signals $x_1(t)$ & $x_2(t)$ can be defined with the help of average form of correlation i.e

Cross correlation of periodic signals:

$$R_{12}(\tau) = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T x_2(t) x_1^*(t-\tau) dt.$$

$$R_{21}(\tau) = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T x_2(t) x_1^*(t-\tau) dt.$$

Two periodic signals $x_1(t)$ & $x_2(t)$ have the same

Time period T_0 , then cross correlation is defined as,

$$R_{12}(\tau) = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x_1(t) x_2^*(t-\tau) dt$$

second cross correlation function is defined as,

$$R_{21}(\tau) = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x_2(t) x_1^*(t-\tau) dt$$

State & Explain the properties of cross correlation function for energy signals.

Ans:

property 1: The cross correlation functions exhibits conjugate symmetry i.e.

$$R_{12}(t) = R_{21}^*(t)$$

that is unlike convolution, cross correlation is not is general commutative i.e.,

~~that is unlike~~ $R_{12}(t) \neq R_{21}(t)$

property 2: If $R_{12}(0) = 0$ i.e.,

$$\int_{-\infty}^{\infty} x_1(t) x_2^*(t) dt = 0$$

then the signals are said to be orthogonal over the entire time interval.

property 3: The cross correlation of the two energy signals corresponds to the multiplication of the fourier transform of one signal & complex conjugate of fourier transform of other signal ; both of them are evaluated at the harmonics of fundamental frequency. i.e.,

$$R_{12}(t) \longleftrightarrow \frac{1}{T_0^2} \sum_{k=-\infty}^{\infty} X_1(kf_0) X_2^*(kf_0) \delta(f - kf_0)$$

property 4: If cross correlation is excited at origin

$$\lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T x_1(t) x_2^*(t) dt = 0$$

$R_{12}(0) = 0$

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property 4: The cross correlation also exhibits conjugate symmetry i.e.

$$R_{12}(\tau) = R_{21}^*(\tau).$$

property 5: Unlike convolution, the cross correlation is not commutative. i.e

$$R_{12}(\tau) \neq R_{21}(\tau).$$

P) prove $R_{12}(\tau) = R_{21}^*(\tau)$.

→ The cross correlation of two signals is given by

$$R_{12}(\tau) = \int_{-\infty}^{\infty} x_1(t) x_2^*(t-\tau) dt.$$

Let $t-\tau = n$ in above Eqn

$$R_{12}(\tau) = \int_{-\infty}^{\infty} x_1(n+\tau) x_2^*(n) dn$$

$$R_{21}(\tau) = \int_{-\infty}^{\infty} x_2(t) x_1^*(t-\tau) dt.$$

Let $t=0$

$$R_{21}(\tau) = \int_{-\infty}^{\infty} x_2(n) x_1^*(n-\tau) dn$$

$$R_{21}^*(\tau) = \int_{-\infty}^{\infty} x_2(n) x_1^*(n+\tau) dn$$

Energy density spectrum (EDS)

→ The distribution of Energy with respect to frequency content of the signal is called Energy spectral density

→ Energy spectral density & autocorrelation function forms a Fourier transform pair

Define the Energy density of a signal.

→

The spectral density functions of the periodic or non-periodic signal $x(t)$ represents the distribution of power or energy in the frequency domain. In other words for a Energy or power signal, the total area under the spectral density curve plotted as a function of frequency is equal to total Energy or average power of the signal.

ESD is given as,

$$\psi(f) = |x(f)|^2$$

$$x(t) \xleftrightarrow{FT} x(f)$$

$x(f)$ is Fourier transform of the signal $x(t)$ & $\psi(f)$ gives the ESD of the signal $x(t)$.

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power spectrum density (PSD)

* The distribution of power of a signal at various frequencies in frequency domain is called PSD.

power spectrum density:

$$S(f) = \frac{1}{T} \sum_{k=-\infty}^{\infty} |x(kT_0)|^2 \delta(f - kT_0)$$